Section 9.6 The Ratio and Root Tests

- **Use the Ratio Test to determine whether a series converges or diverges.**
- **Use the Root Test to determine whether a series converges or diverges.**
- **Review the tests for convergence and divergence of an infinite series.**

Writing a Series One of the following conditions guarantees that a series will diverge, two conditions guarantee that a series will converge, and one has no guarantee—the series can either converge or diverge. Which is which? Explain your reasoning.

a. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$

b. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}$

c. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

d. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2$

The Ratio Test

This section begins with a test for absolute convergence—the **Ratio Test.**

THEOREM 9.17 Ratio Test

Let $\Sigma\, a_n$ be a series with nonzero terms.

1. $\Sigma\, a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\Sigma\, a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Proof To prove Property 1, assume that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r < 1$$

and choose R such that $0 \le r < R < 1$. By the definition of the limit of a sequence, there exists some $N > 0$ such that $|a_{n+1}/a_n| < R$ for all $n > N$. Therefore, you can write the following inequalities.

$$\begin{aligned}
|a_{N+1}| &< |a_N|R\\
|a_{N+2}| &< |a_{N+1}|R < |a_N|R^2\\
|a_{N+3}| &< |a_{N+2}|R < |a_{N+1}|R^2 < |a_N|R^3\\
&\vdots
\end{aligned}$$

The geometric series $\Sigma\, |a_N|R^n = |a_N|R + |a_N|R^2 + \cdots + |a_N|R^n + \cdots$ converges, and so, by the Direct Comparison Test, the series

$$\sum_{n=1}^{\infty} |a_{N+n}| = |a_{N+1}| + |a_{N+2}| + \cdots + |a_{N+n}| + \cdots$$

also converges. This in turn implies that the series $\Sigma\, |a_n|$ converges, because discarding a finite number of terms ($n = N - 1$) does not affect convergence. Consequently, by Theorem 9.16, the series $\Sigma\, a_n$ converges absolutely. The proof of Property 2 is similar and is left as an exercise (see Exercise 98).

NOTE The fact that the Ratio Test is inconclusive when $|a_{n+1}/a_n| \to 1$ can be seen by comparing the two series $\Sigma\,(1/n)$ and $\Sigma\,(1/n^2)$. The first series diverges and the second one converges, but in both cases

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

Although the Ratio Test is not a cure for all ills related to tests for convergence, it is particularly useful for series that *converge rapidly*. Series involving factorials or exponentials are frequently of this type.

EXAMPLE 1 Using the Ratio Test

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Solution Because $a_n = 2^n/n!$, you can write the following.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!}\right] \\
&= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right] \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0
\end{aligned}$$

Therefore, the series converges.

STUDY TIP A step frequently used in applications of the Ratio Test involves simplifying quotients of factorials. In Example 1, for instance, notice that

$$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}.$$

EXAMPLE 2 Using the Ratio Test

Determine whether each series converges or diverges.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

a. This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[(n+1)^2\left(\frac{2^{n+2}}{3^{n+1}}\right)\left(\frac{3^n}{n^2 2^{n+1}}\right)\right] \\
&= \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2} \\
&= \frac{2}{3} < 1
\end{aligned}$$

b. This series diverges because the limit of $|a_{n+1}/a_n|$ is greater than 1.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{(n+1)!}\left(\frac{n!}{n^n}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{(n+1)}\left(\frac{1}{n^n}\right)\right] \\
&= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \\
&= e > 1
\end{aligned}$$

EXAMPLE 3 A Failure of the Ratio Test

Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$.

Solution The limit of $|a_{n+1}/a_n|$ is equal to 1.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\left(\frac{\sqrt{n+1}}{n+2}\right)\left(\frac{n+1}{\sqrt{n}}\right)\right] \\
&= \lim_{n\to\infty}\left[\sqrt{\frac{n+1}{n}}\left(\frac{n+1}{n+2}\right)\right] \\
&= \sqrt{1}(1) \\
&= 1
\end{aligned}$$

The Ratio Test is also inconclu- r any p-series.

So, the Ratio Test is inconclusive. To determine whether the series converges, you need to try a different test. In this case, you can apply the Alternating Series Test. To show that $a_{n+1} \le a_n$, let

$$f(x) = \frac{\sqrt{x}}{x+1}.$$

Then the derivative is

$$f'(x) = \frac{-x+1}{2\sqrt{x}(x+1)^2}.$$

Because the derivative is negative for $x > 1$, you know that f is a decreasing function. Also, by L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{x}}{x+1} &= \lim_{x\to\infty}\frac{1/(2\sqrt{x})}{1} \\
&= \lim_{x\to\infty}\frac{1}{2\sqrt{x}} \\
&= 0.
\end{aligned}$$

Therefore, by the Alternating Series Test, the series converges.

The series in Example 3 is *conditionally convergent.* This follows from the fact that the series

$$\sum_{n=1}^{\infty} |a_n|$$

diverges (by the Limit Comparison Test with $\Sigma\, 1/\sqrt{n}$), but the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

TECHNOLOGY A computer or programmable calculator can reinforce the conclusion that the series in Example 3 converges *conditionally.* By adding the first 100 terms of the series, you obtain a sum of about -0.2. (The sum of the first 100 terms of the series $\Sigma\, |a_n|$ is about 17.)

The Root Test

The next test for convergence or divergence of series works especially well for series involving nth powers. The proof of this theorem is similar to that given for the Ratio Test, and is left as an exercise (see Exercise 99).

THEOREM 9.18 Root Test

Let $\Sigma\, a_n$ be a series.

1. $\Sigma\, a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. $\Sigma\, a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

EXAMPLE 4 Using the Root Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.$$

Solution You can apply the Root Test as follows.

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} \\
&= \lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}} \\
&= \lim_{n\to\infty} \frac{e^2}{n} \\
&= 0 < 1
\end{aligned}$$

NOTE The Root Test is always inconclusive for any p-series.

Because this limit is less than 1, you can conclude that the series converges absolutely (and therefore converges).

FOR FURTHER INFORMATION For more information on the usefulness of the Root Test, see the article "$N!$ and the Root Test" by Charles C. Mumma II in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com.*

To see the usefulness of the Root Test for the series in Example 4, try applying the Ratio Test to that series. When you do this, you obtain the following.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{e^{2(n+1)}}{(n+1)^{n+1}} \div \frac{e^{2n}}{n^n}\right] \\
&= \lim_{n\to\infty} e^2 \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} e^2 \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) \\
&= 0
\end{aligned}$$

Note that this limit is not as easily evaluated as the limit obtained by the Root Test in Example 4.

Strategies for Testing Series

You have now studied 10 tests for determining the convergence or divergence of an infinite series. (See the summary in the table on page 644.) Skill in choosing and applying the various tests will come only with practice. Below is a set of guidelines for choosing an appropriate test.

Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test.

EXAMPLE 5 Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$ **e.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ **f.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Solution

a. For this series, the limit of the nth term is not 0 $\left(a_n \to \frac{1}{3} \text{ as } n \to \infty\right)$. So, by the nth-Term Test, the series diverges.

b. This series is geometric. Moreover, because the ratio $r = \pi/6$ of the terms is less than 1 in absolute value, you can conclude that the series converges.

c. Because the function $f(x) = xe^{-x^2}$ is easily integrated, you can use the Integral Test to conclude that the series converges.

d. The nth term of this series can be compared to the nth term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.

e. This is an alternating series whose nth term approaches 0. Because $a_{n+1} \le a_n$, you can use the Alternating Series Test to conclude that the series converges.

f. The nth term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

g. The nth term of this series involves a variable that is raised to the nth power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges.

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Exercises for Section 9.6

In Exercises 1–4, verify the formula.

1. $\dfrac{(n+1)!}{(n-2)!} = (n+1)(n)(n-1)$

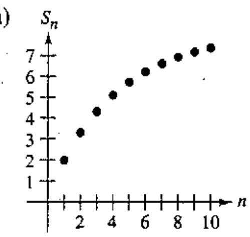
2. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{1}{(2k)(2k-1)}$

3. $1 \cdot 3 \cdot 5 \cdots (2k-1) = \dfrac{(2k)!}{2^k k!}$

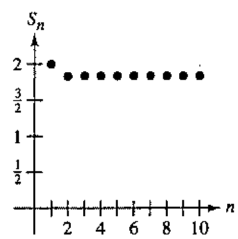
4. $\dfrac{1}{1 \cdot 3 \cdot 5 \cdots (2k-5)} = \dfrac{2^k k!(2k-3)(2k-1)}{(2k)!}, \quad k \geq 3$

In Exercises 5–10, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

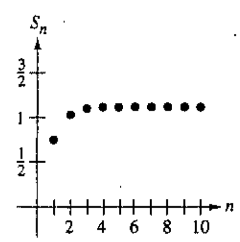
(a)



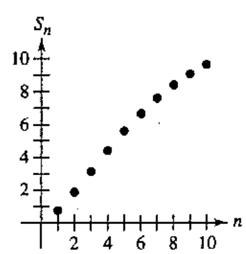
(b)



(c)



(d)



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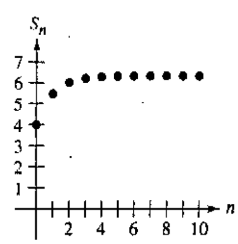
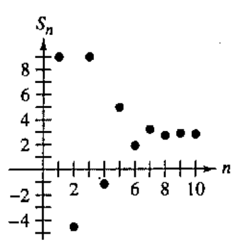


(f)



5. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n\left(\frac{1}{n!}\right)$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!}$

9. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$

10. $\displaystyle\sum_{n=0}^{\infty} 4e^{-n}$

Numerical, Graphical, and Analytic Analysis **In Exercises 11 and 12, (a) verify that the series converges. (b) Use a graphing utility to find the indicated partial sum S_n and complete the table. (c) Use a graphing utility to graph the first 10 terms of the sequence of partial sums. (d) Use the table to estimate the sum of the series. (e) Explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.**

n	5	10	15	20	25
S_n					

11. $\displaystyle\sum_{n=1}^{\infty} n^2\left(\frac{5}{8}\right)^n$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n!}$

In Exercises 13–32, use the Ratio Test to determine the convergence or divergence of the series.

13. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

14. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$

15. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n!}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^n+1}$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

31. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n[2 \cdot 4 \cdot 6 \cdots (2n)]}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

In Exercises 33–36, verify that the Ratio Test is inconclusive for the p-series.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

In Exercises 37–50, use the Root Test to determine the convergence or divergence of the series.

37. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

38. $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

39. $\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^n$

40. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{2n-1}\right)^n$

41. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

42. $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

43. $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$

44. $\sum_{n=0}^{\infty} e^{-n}$

45. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

46. $\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

47. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

48. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

49. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

50. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

In Exercises 51–68, determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n}$

52. $\sum_{n=1}^{\infty} \frac{5}{n}$

53. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

54. $\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

55. $\sum_{n=1}^{\infty} \frac{2n}{n+1}$

56. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

57. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n}$

58. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

59. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

60. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

61. $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

62. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

63. $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$

64. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

65. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

66. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

67. $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

68. $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n(2n-1)n!}$

In Exercises 69–72, identify the two series that are the same.

69. (a) $\sum_{n=1}^{\infty} \frac{n5^n}{n!}$
(b) $\sum_{n=0}^{\infty} \frac{n5^n}{(n+1)!}$
(c) $\sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!}$

70. (a) $\sum_{n=4}^{\infty} n\left(\frac{3}{4}\right)^n$
(b) $\sum_{n=0}^{\infty} (n+1)\left(\frac{3}{4}\right)^n$
(c) $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$

71. (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$
(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$
(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!}$

72. (a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}}$
(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$
(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)2^n}$

In Exercises 73 and 74, write an equivalent series with the index of summation beginning at $n = 0$.

73. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

74. $\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!}$

In Exercises 75 and 76, (a) determine the number of terms required to approximate the sum of the series with an error less than 0.0001, and (b) use a graphing utility to approximate the sum of the series with an error less than 0.0001.

75. $\sum_{k=1}^{\infty} \frac{(-3)^k}{2^k k!}$

76. $\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$

In Exercises 77–82, the terms of a series $\sum_{n=1}^{\infty} a_n$ are defined recursively. Determine the convergence or divergence of the series. Explain your reasoning.

77. $a_1 = \frac{1}{2}, a_{n+1} = \frac{4n-1}{3n+2}a_n$

78. $a_1 = 2, a_{n+1} = \frac{2n+1}{5n-4}a_n$

79. $a_1 = 1, a_{n+1} = \frac{\sin n + 1}{\sqrt{n}}a_n$

80. $a_1 = \frac{1}{5}, a_{n+1} = \frac{\cos n + 1}{n}a_n$

81. $a_1 = \frac{1}{3}, a_{n+1} = \left(1 + \frac{1}{n}\right)a_n$

82. $a_1 = \frac{1}{4}, a_{n+1} = \sqrt[n]{a_n}$

In Exercises 83–86, use the Ratio Test or the Root Test to determine the convergence or divergence of the series.

83. $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$

84. $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \frac{5}{3^4} + \frac{6}{3^5} + \cdots$

85. $\frac{1}{(\ln 3)^3} + \frac{1}{(\ln 4)^4} + \frac{1}{(\ln 5)^5} + \frac{1}{(\ln 6)^6} + \cdots$

86. $1 + \frac{1 \cdot 3}{1 \cdot 2 \cdot 3} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots$

In Exercises 87–92, find the values of x for which the series converges.

87. $\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$

88. $\sum_{n=0}^{\infty} \left(\frac{x+1}{4}\right)^n$

89. $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n}$

90. $\sum_{n=0}^{\infty} 2(x-1)^n$

91. $\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$

92. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

Writing About Concepts

93. State the Ratio Test.

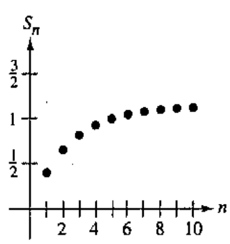
94. State the Root Test.

95. You are told that the terms of a positive series appear to approach zero rapidly as n approaches infinity. In fact, $a_7 \le 0.0001$. Given no other information, does this imply that the series converges? Support your conclusion with examples.

96. The graph shows the first 10 terms of the sequence of partial sums of the convergent series

$$\sum_{n=1}^{\infty} \left(\frac{2n}{3n+2}\right)^n.$$

Find a series such that the terms of its sequence of partial sums are less than the corresponding terms of the sequence in the figure, but such that the series diverges. Explain your reasoning.



97. Using the Ratio Test, it is determined that an alternating series converges. Does the series converge conditionally or absolutely? Explain.

98. Prove Property 2 of Theorem 9.17.

99. Prove Theorem 9.18. (*Hint for Property 1:* If the limit equals $r < 1$, choose a real number R such that $r < R < 1$. By the definitions of the limit, there exists some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for $n > N$.)

100. Show that the Root Test is inconclusive for the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

101. Show that the Ratio Test and the Root Test are both inconclusive for the logarithmic p-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

102. Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$$

when (a) $x = 1$, (b) $x = 2$, (c) $x = 3$, and (d) x is a positive integer.

103. Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

104. ***Writing*** Read the article "A Differentiation Test for Absolute Convergence" by Yaser S. Abu-Mostafa in *Mathematics Magazine*. Then write a paragraph that describes the test. Include examples of series that converge and examples of series that diverge.

Putnam Exam Challenge

105. Is the following series convergent or divergent?

$$1 + \frac{1}{2}\cdot\frac{19}{7} + \frac{2!}{3^2}\left(\frac{19}{7}\right)^2 + \frac{3!}{4^3}\left(\frac{19}{7}\right)^3 + \frac{4!}{5^4}\left(\frac{19}{7}\right)^4 + \cdots$$

106. Show that if the series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

converges, then the series

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} + \cdots$$

converges also.